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On existence of noncritical vertices in digraphs

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Abstract

Let D be a strongly connected digraphs on $n \geq 4$ vertices. A vertex v of D is noncritical, if the digraph $D-v$ is strongly connected. We prove, that if sum of the degrees of any two adjacent vertices of D is at least $n+1$, then there exists a noncritical vertex in D, and if sum of the degrees of any two adjacent vertices of D is at least $n+2$, then there exist two noncritical vertices in D. A series of examples confirm that these bounds are tight.

1 Introduction

In this paper we consider a digraph D without loops and multiple arcs, i.e. any two vertices x and y are connected by at most two arcs (at most one arc of each direction).

For a subgraph T of the digraph D we denote by $V(T)$ the set of vertices of T and by $\overline{V(T)}$ the set of vertices of D which do not belong to the subgraph T. We denote by $T-x$ the subgraph obtained from T by deleting the vertex x and all arcs incident to x.

Definition 1. A digraph is called *strongly connected* if for any two its vertices x, y there is a path from x to y.

Definition 2. A vertex v of a strongly connected digraph D is called *noncritical*, if the digraph $D-v$ is strongly connected.

Definition 3. *The degree* of a vertex x in a digraph D (notation: $deg(x)$) is the number of vertices adjacent to x. (In the case where vertices x and y are connected by several arcs we count the vertex y once.)

S. V. Savchenko [3] in 2006 has proved that a strongly connected digraph D with n vertices and vertex degrees at least $\frac{3n}{4}$ has two noncritical vertices. There are no further results on this subject.

It follows from our main result, that if minimal vertex degree of a digraph is at least $\frac{n+1}{2}$ then this digraph has a noncritical vertex, and if minimal vertex degree of a digraph is at least $\frac{n+1}{2}$ then this digraph has two noncritical vertices.

As well as the author of [3], we need the following lemma, formulated in [1].

Lemma 1. *Let D be a strongly connected digraph and S be a proper strongly connected subgraph of D. Then S is a maximal proper strongly connected subgraph of D if and only if the three following conditions hold:*

1) *There exists a vertex $\omega_{in} \in \overline{V(S)}$, such that any arc from $V(S)$ to $\overline{V(S)}$ ends in the vertex ω_{in};*

2) *There exists a vertex $\omega_{out} \in \overline{V(S)}$, such that any arc from $\overline{V(S)}$ to $V(S)$ begins at the vertex ω_{out};*

3) *There exists a unique simple path from ω_{in} to ω_{out} in D and this path contains all vertices of the set $\overline{V(S)}$ and only them (see figure 1).*

2 Search for noncritical vertices

Theorem 1. *Let D be a strongly connected digraph with $n \geqslant 4$ vertices, such that for any two adjacent vertices x and y of this digraph the inequality $deg(x)+deg(y) \geqslant n+1$ holds. Then D has a noncritical vertex.*

Proof. We consider two cases A and B: in case A the minimal vertex degree of D is at least 3, and in case B there is a vertex of degree less than 3 in D.

A. *The minimal vertex degree of D is at least* 3.

Consider a maximal proper strongly connected subgraph S of the digraph D.

1° *If $\overline{V(S)}$ consists of one vertex*, then this vertex is noncritical and the theorem is proved.

Hence in what follows the set $\overline{V(S)}$ consists of at least two vertices, then the vertices ω_{in} and ω_{out} are different.

2° *Let $\overline{V(S)}$ contains at least four vertices.*

Consider a path from ω_{in} to ω_{out}, which contains all vertices of the set $\overline{V(S)}$ and only them. In this case we can choose two successive vertices a_1 and a_2 in this path, which are different from ω_{in} and ω_{out} (see figure 1).

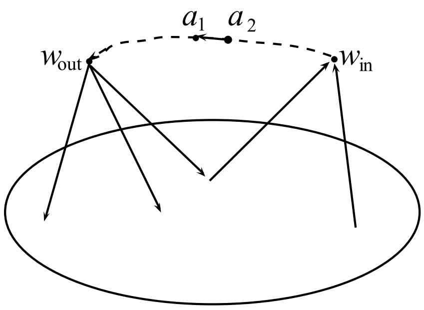

Figure 1:

Then there are no arcs between $\{a_1, a_2\}$ and S, hence the degree of both vertices a_1 and a_2 is at most $|\overline{V(S)}| - 1$. Consider two adjacent vertices $b_1, b_2 \in V(S)$. (These vertices exist since otherwise S consists of one vertex and this vertex has degree at most 2.) The degree of each of vertices b_1, b_2 is not more than $|V(S)| - 1 + 2 = |V(S)| + 1$ (the vertex b_i can be adjacent to the vertices of S different from b_i and to ω_{in}, ω_{out}).

We have chosen two pairs of adjacent vertices, hence, we obtain the following inequality:

$$2(n+1) \leqslant deg(a_1) + deg(a_2) + deg(b_1) + deg(b_2) \leqslant$$
$$2(|\overline{V(S)}| - 1) + 2(|V(S)| + 1) = 2|V(D)| = 2n,$$

that is impossible.

3° *Let $\overline{V(S)}$ contains three vertices.*

In this case a vertex $a \in \overline{V(S)}$, which is different from ω_{in} and ω_{out}, is adjacent only to ω_{in} and ω_{out}, i.e. $deg(a) = 2$. We obtain a contradiction.

4° *Let $\overline{V(S)}$ contains two vertices*, i.e. $\overline{V(S)} = \{\omega_{in}, \omega_{out}\}$.

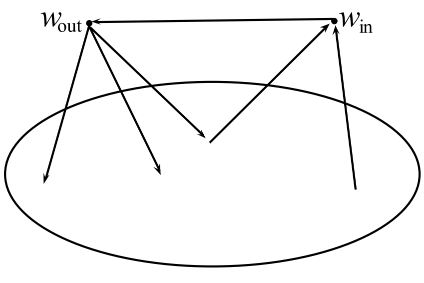

Figure 2:

Let us construct an *incoming tree* T_{in} of the vertex ω_{in} in the subgraph $D - \omega_{out}$. The vertex ω_{in} is *the root* of this tree. Level 1

consists of the vertices which have outgoing arcs to ω_{in}, and so on: level k consists of the vertices which do not belong to previous levels and have outgoing arcs to vertices of level $(k-1)$. It follows from the strong connectivity of D, that for any vertex $x \in S$ there is a path from x to ω_{in} in D. Hence, any vertex of the set S belongs to some level. For each vertex, with the exception of ω_{in}, we draw exactly one outgoing arc to a vertex of previous level (see figure 3).

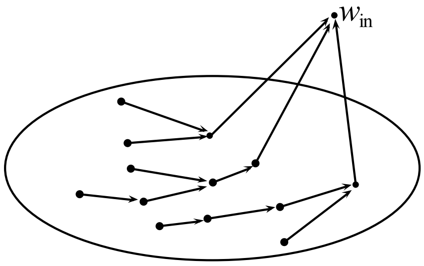

Figure 3: Incoming tree.

Now consider T_{in} as an undirected tree. Let us call any connected component of $T_{in} - \omega_{in}$ a *branch*. Note, that every branch contains a leaf of this tree, and the number of branches is equal to $deg(\omega_{in}) - 1$, since T_{in} contains all arcs incident to ω_{in}, with the exception of arcs between ω_{in} and ω_{out}. Denote by A_{in} the set of all leaves of T_{in}. We have proved that $|A_{in}| \geqslant deg(\omega_{in}) - 1$. Clearly, for any vertex $x \in A_{in}$, there is a path from any other vertex of the set S to ω_{in} in $T_{in} - x$.

Similarly, we construct an *outgoing tree* of the vertex ω_{out} in the subgraph $D - \omega_{in}$ and the set A_{out}. For any vertex $x \in A_{out}$, there is a path from ω_{out} to any other vertex of the set S in $T_{out} - x$. Similarly, $|A_{out}| \geq deg(\omega_{out}) - 1$. The vertices ω_{in} and ω_{out} are adjacent, whence it follows:

$$|A_{in}| + |A_{out}| \geqslant deg(\omega_{in}) - 1 + deg(\omega_{out}) - 1 \geqslant n + 1 - 2 = n - 1.$$

Since $|A_{in} \cup A_{out}| \leq |S| = n - 2$, there exists a vertex $x \in A_{in} \cap A_{out}$.

For any vertex $y \in S \setminus x$ there is a path from y to ω_{in} and a path from ω_{out} to y in the graph $D - x$. Since there is an arc from ω_{in} to ω_{out}, the graph $D - x$ is strongly connected and the vertex x is noncritical.

B. *There is a vertex of degree less than* 3 *in* D.

Let q is a vertex of degree less than 3. Clearly, there exists a vertex p_1 adjacent to q. We know, that

$$deg(q) + n - 1 \geqslant deg(q) + deg(p_1) \geqslant n + 1,$$

hence we obtain $deg(q) = 2$ and $deg(p_1) = n - 1$. Thus there exists another vertex p_2 adjacent to q. Similarly, $deg(p_2) = n - 1$.

Since $deg(p_1) = deg(p_2) = n - 1$, there exists an arc between p_1 and p_2 (maybe two arcs of different directions). Without loss of generality assume that there is an arc p_1p_2 in D. We assume that q is not a noncritical vertex (otherwise the theorem is proved). Then the graph $D - q$ is not strongly connected, hence there are two vertices x and y, such that any path from x to y in D contains the vertex q. Consider the shortest path P from x to y. Clearly, P must pass both vertices p_1 and p_2. Since P is the shortest path, it passes the arcs p_2q and qp_1 (otherwise the shortest path must pass the arc p_1p_2 and avoid q).

Hence there is an oriented cycle qp_1p_2 in the digraph D. Since this cycle does not contain all vertices of D, there is a maximal strongly connected proper subgraph S, which contains q, p_1 and p_2. If $|\overline{V(S)}| > 2$, then there exists a vertex $a \in \overline{S}$, different from $\omega_{in}, \omega_{out}$. Clearly, there is no arc between p_1 and a, i.e. $deg(p_1) < n - 1$, we obtain a contradiction. The remaining cases $|\overline{V(S)}| = 1$ and $|\overline{V(S)}| = 2$ are similar to the subcases 1° and 4° of case A. (It does not matter whether there are vertices of degrees 1 and 2 or not in the proofs of these subcases). $\square$

Corollary 1. *Let D be a strongly connected digraph with $n \geqslant 4$ vertices and vertex degrees at least $\frac{n+1}{2}$. Then D has a noncritical vertex.*

Remark 1. 1) The bounds $n+1$ in theorem 1 and $\frac{n+1}{2}$ in corollary 1 are tight. Let us construct for an even n a graph D (see figure 4), such that:

— $V(D) = \{a_1, \dots a_{n/2}, b_1, \dots b_{n/2}\}$;

— $E(D)$ consists of arcs a_ib_i and arcs b_ia_j, where $i \neq j$.

Clearly, all vertex degrees in this graph are equal to $\frac{n}{2}$ and there are no noncritical vertices.

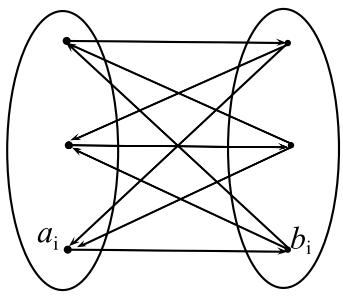

Figure 4: A graph without noncritical vertices.

2) The bound $n+1$ in theorem 1 is also tight for odd n. Let us construct a digraph D which is suitable for all $n \geqslant 6$ (see figure 5):

— $V(D) = \{a_1, \dots, a_{n-4}, x_1, \dots, x_4\}$.

— $E(D)$ consists of arcs $a_i a_{i+1}$, $a_j a_i$ $(j > i+1)$, $a_i x_3$, $x_2 a_i$, $a_{n-4} x_1$, $x_1 x_2$, $x_2 x_3$, $x_3 x_4$, $x_4 a_1$ (where $i, j \in \{1, \dots, n-4\}$).

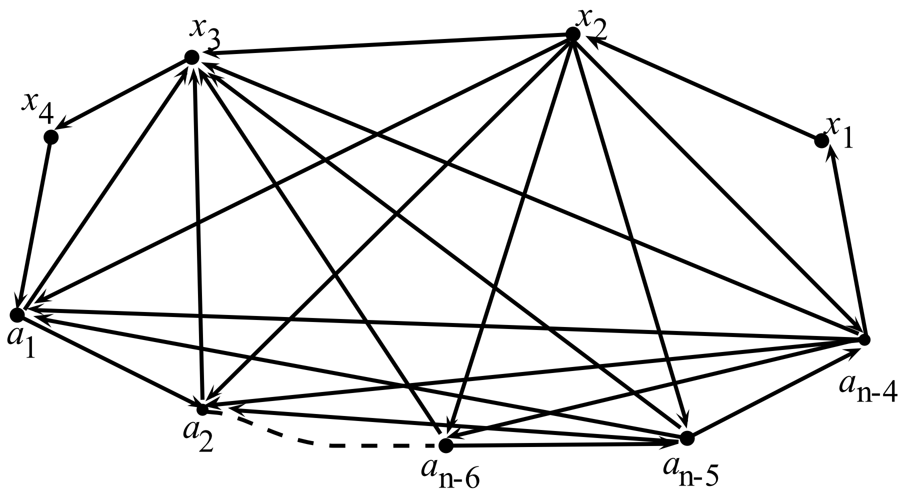

Figure 5: A graph without noncritical vertices.

Clearly, the digraph D is strongly connected. Let us verify the condition on sum of degrees of pairs of adjacent vertices. The vertices x_1 and x_4 have degree 2, the vertices a_1, a_{n-4}, x_2, x_3 have degree $n-2$, all other vertices have degree $n-3$. Each vertex of degree 2 is adjacent only to vertices of degree $n-2$, i.e. the sum of degrees of these pairs is n. Any other vertex has degree at least $n-3$, hence, any other pair of adjacent vertices has sum of degrees at least $2(n-3) = 2n-6 \geqslant n$.

Let us assure, that D has no noncritical vertices. In the graph $D - a_i$ (where $i > 1$) there is no path from $\{x_3, x_4, a_1, \dots, a_{i-1}\}$ to x_1. In the graph $D - x_1$ there are no arcs with the end at x_2. In the graph $D - x_2$ there are no arcs with the beginning at x_2. In the graph $D - x_3$ there are no arcs with the end at x_4. In the graph $D - x_4$ there are no arcs with the beginning at x_3. In the graph $D - a_1$ there are no arcs with the beginning at x_4. Hence, there are no noncritical vertices in D.

Corollary 2. *Let D be a strongly connected digraph with $n \geqslant 4$ vertices, such that for any two adjacent vertices x and y of this digraph the inequality $deg(x) + deg(y) \geqslant n+2$ holds. Then D has two noncritical vertices.*

Proof. We claim, that the degree of any vertex in the graph D is at least 3. (Otherwise, if there is a vertex x of degree at most two, then a vertex adjacent to x must have degree at least n. Clearly, that is impossible). Hence, we have the case A of theorem 1. Consider two cases.

1° *For any maximal strongly connected proper subgraph S we have $|\overline{V(S)}| = 1$.*

By theorem 1 there exists a noncritical vertex x_1 in D. Consider a maximal strongly connected proper subgraph which contains x_1. Let it does not contain x_2. Then x_1 and x_2 are two different noncritical vertices.

2° *There exists a maximal strongly connected subgraph S, such that $|\overline{V(S)}| \geqslant 2$.*

Then by the reasonings of theorem 1 only the case where $|\overline{V(S)}| = 2$ is possible. Consider the sets A_{in} and A_{out} constructed in the proof of theorem 1 (see subcase 4° of case A). Now we have the inequality

$$|A_{in}| + |A_{out}| \geqslant deg(\omega_{in}) - 1 + deg(\omega_{out}) - 1 \geqslant n + 2 - 2 = n.$$

Since $|A_{in} \cup A_{out}| \leq |S| = n - 2$, there exist at least two vertices in $A_{in} \cap A_{out}$. Clearly, these vertices are noncritical. $\square$

Corollary 3. *Let D be a strongly connected digraph with $n \geqslant 4$ vertices and vertex degrees at least $\frac{n+2}{2}$. Then D has at least two noncritical vertices.*

Remark 2. 1) The bounds $n+2$ in corollary 2 and $\frac{n+2}{2}$ in corollary 3 are tight. Let us construct for an odd n a graph D (see figure 6), such that:

— $V(D) = \{a_1, \dots a_{(n-1)/2}, b_1, \dots b_{(n-1)/2}, x\}$;

— $E(D)$ consists of arcs a_ib_i, arcs b_ia_j, where $i \neq j$, arcs xa_i and arcs b_ix.

Clearly, all vertex degrees in this graph are at least $\frac{n+1}{2}$ and x is the only noncritical vertex.

2) The bound $n + 2$ in corollary 2 is also tight for even n. Let us construct a digraph D which is suitable for all $n \geqslant 5$ (see figure 7):

— $V(D) = \{a_1, \dots, a_{n-1}, x\}$.

— $E(D)$ consists of arcs a_ia_{i+1}, a_ja_i $(j > i + 1)$, xa_1, $a_{n-1}x$ (where $i, j \in \{1, \dots, n - 1\}$).

Clearly, the digraph D is strongly connected. Let us verify the condition on sum of degrees of pairs of adjacent vertices. The vertex x of degree 2 is adjacent in D only to vertices of degree $n - 1$, i.e. the sum of degrees of these pairs is $n + 1$. Any other vertex has degree

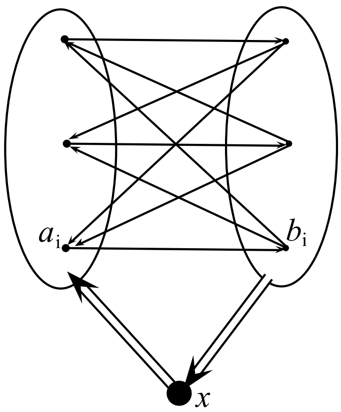

Figure 6: A graph with only one noncritical vertex.

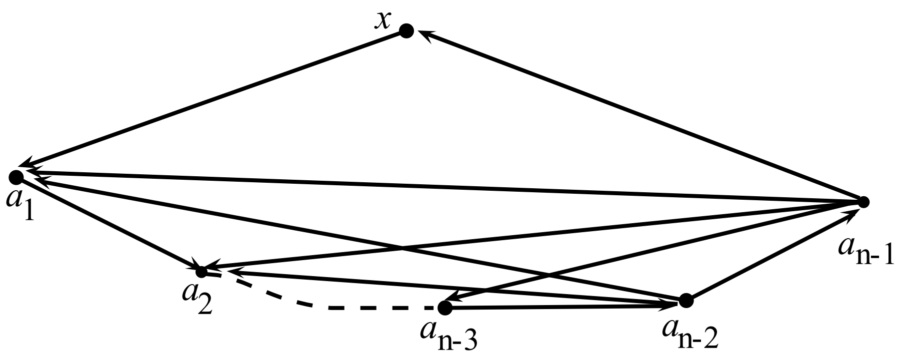

Figure 7: A graph with only one noncritical vertex.

at least $n-2$, hence, any other pair of adjacent vertices has sum of degrees at least $2(n-2) = 2n-4 \geqslant n+1$.

Let us assure, that x is the only noncritical vertex in D. Clearly, in the graph $D - a_i$ (where $i > 1$) there is no path from $\{a_1, \ldots, a_{i-1}\}$ to x. In the graph $D - a_1$ there are no arcs with beginning at x.

Remark 3. The proof of theorem 1 in the case $|\overline{V(S)}| > 1$ and minimal vertex degree is d gives us $2d - n$ noncritical vertices. In spite of this, no lower bound on minimal vertex degree can provide three noncritical vertices in a strongly connected digraph D.

For any n there exists a strong tournament (i.e. a strongly connected digraph in which each pair of vertices is connected by exactly one arc) with precisely two noncritical vertices. Let us construct this tournament on n vertices $a_1, \ldots, a_n$ (see figure 8). This tournament has arcs of type $a_i a_{i+1}$ (where $i < n$), and of type $a_i a_j$ for each pair i, j, where $i > j + 1$.

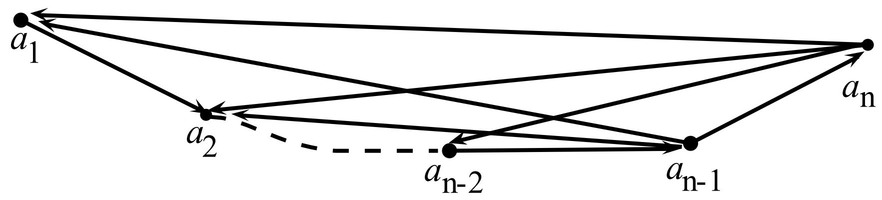

Figure 8: Strong tournament with two noncritical vertices.

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